CHAPTER 4

OPEN STUDENT FOUNDATION
Std 12 : MATHS
PRACTICE SHEET DAY 3

Date : 21/02/24

Section A

● **Write the answer of the following questions. [Each carries 2 Marks]** **[12]**

1. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ then show that $|3A| = 27|A|$.

2. Find value of k if area of triangle is 4 sq. units and vertices are : $(k, 0)$ $(4, 0)$, $(0, 2)$

3. Find area of the triangle with vertice at the point given in each of the following : (2, 7), (1, 1), (10, 8)

4. Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

5. Examine the consistency of the system of linear equation in :
$x + 3y = 5$
$2x + 6y = 8$

6. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that $(A^{-1})^{-1} = A$.

Section B

● **Write the answer of the following questions. [Each carries 3 Marks]** **[6]**

7. Write Minors and Cofactors of the element of following determinant : $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

8. Verify $A\ (adj\ A) = (adj\ A)\ A = |A|\ I$: $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

Section C

● **Write the answer of the following questions. [Each carries 4 Marks]** **[32]**

9. Find the inverse of the following matric : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

10. Find the inverse of the following matric : $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

11. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

12. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, Show that $A^3 - 6A^2 + 5A + 11\ I = O$. Hence, find A^{-1}.

13. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ Verify the result $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}.

14. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ then find $(AB)^{-1}$.

15. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that (i) $[adj\ A]^{-1} = adj\ (A^{-1})$ (ii) $(A^{-1})^{-1} = A$.

16. Solve the following system of linear equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Section A

- **Write the answer of the following questions. [Each carries 2 Marks]** **[12]**

1. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ then show that $|3A| = 27|A|$.

➥ $|A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix} = 1(4 - 0) = 4$

$27\,|A| = 27(4) = 108$

$3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$

$\therefore |3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix} = 3(36 - 0) = 108$

$\therefore\ |3\,A| = 27\,|A| = 108$

2. Find value of k if area of triangle is 4 sq. units and vertices are : $(k, 0)$ $(4, 0)$, $(0, 2)$

➥ **$(k, 0)$ $(4, 0)$, $(0, 2)$**

$D = \begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$

$= k(0 - 2) + 1(8)$

$\therefore\ D = -2k + 8$

Now area of the triangle $= \frac{1}{2}|D|$

$\therefore\ 4 = \frac{1}{2}|-2k + 8|$

$\therefore\ 4 = |-k + 4|$

$-k + 4 = 4$ **or** $-k + 4 = -4$

$\therefore\ k = 0,$ **or** $k = 8$

$\therefore$ The values of k are 0 **or** 8.

3. Find area of the triangle with vertice at the point given in each of the following : (2, 7), (1, 1), (10, 8)

➥ Let A(2, 7), B(1, 1) and C(10, 8) are vertices of ΔABC.

$\therefore$ Area of the triangle $= \frac{1}{2}|D|,$

Where $D = \begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$

$\therefore$ D $= 2(1 - 8) - 7(1 - 10) + 1(8 - 10)$

$= 2(-7) - 7(-9) + 1(-2)$

$= -14 + 63 - 2$

$\therefore$ D $= 47$

$\therefore$ Area of the triangle $= \frac{1}{2}|D|$

$= \frac{1}{2}|47| = \frac{47}{2}$ sq.unit.

4. Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

➥ $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Elements of second row are a_{21}, a_{22} and a_{23}.

Cofactor of a_{21} $A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = -(9-16) = 7$

Cofactor of a_{22} $A_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$

Cofactor of a_{23} $A_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = -(10-3) = -7$

$\therefore \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$= 2(7) + 0(7) + 1(-7)$

$= 14 + 0 - 7 = 7$

5. Examine the consistency of the system of linear equation in :

$x + 3y = 5$

$2x + 6y = 8$

➥ Given equations can be written in matrix form as, AX = B.

where, $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0.$

$\therefore A^{-1}$ does not exists.

To find *adj* A $\Rightarrow$

$A_{11} = (-1)^{1+1}\, 6 = 6$ $\quad A_{21} = (-1)^{2+1}\, 3 = -3$

$A_{12} = (-1)^{1+2}\, 2 = -2$ $\quad A_{22} = (-1)^{2+2}\, 1 = 1$

$\therefore$ *adj* A = $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

Now $adj\ A \cdot B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30 - 24 \\ -10 + 8 \end{bmatrix}$

$$= \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ Hence, equations are inconsistent and their solution does not exists.

6. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that $(A^{-1})^{-1} = A$.

➥ Let $B = A^{-1} = -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$

$$\therefore B = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

$$|B| = -\frac{14}{13}\left(-\frac{4}{169} - \frac{9}{169}\right) + \frac{11}{13}\left(-\frac{11}{169} - \frac{15}{169}\right) + \frac{5}{13}\left(-\frac{33}{169} + \frac{20}{169}\right)$$

$$= -\frac{14}{13}\left(-\frac{13}{169}\right) + \frac{11}{13}\left(-\frac{26}{169}\right) + \frac{5}{13}\left(-\frac{13}{169}\right)$$

$$= +\frac{14}{169} - \frac{22}{169} - \frac{5}{169}$$

$$= -\frac{13}{169}$$

$$= -\frac{1}{13}$$

$$adj\ B = adj\begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

$$= \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & 65 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

$$(B^{-1}) = \frac{1}{|B|}\ adj\ B$$

$$= \frac{1}{-\frac{1}{13}} \times \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$= A$$

Hence, $(A^{-1})^{-1} = A$

Section B

- **Write the answer of the following questions. [Each carries 3 Marks]** **[6]**

7. Write Minors and Cofactors of the element of following determinant : $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

➥ $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Minors :	**Cofactors :**
Minor of 1 $= M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$	$A_{11} = (-1)^{1+1}(1) = 1$
Minor of 0 $= M_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$	$A_{12} = (-1)^{1+2}(0) = 0$
Minor of 0 $= M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$	$A_{13} = (-1)^{1+3}(0) = 0$
Minor of 0 $= M_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$	$A_{21} = (-1)^{2+1}(0) = 0$
Minor of 1 $= M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$	$A_{22} = (-1)^{2+2}(1) = 1$
Minor of 0 $= M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$	$A_{23} = (-1)^{2+3}(0) = 0$
Minor of 0 $= M_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$	$A_{31} = (-1)^{3+1}(0) = 0$
Minor of 0 $= M_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$	$A_{32} = (-1)^{3+2}(0) = 0$
Minor of 1 $= M_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$	$A_{33} = (-1)^{3+3}(1) = 1$

8. Verify $A\,(adj\ A) = (adj\ A)\,A = |A|\,I$: $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

➥ $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

$\therefore\ |A| = -12 + 12 = 0$

$A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$A_{11} = (-1)^{1+1}(-6) = -6 \quad A_{12} = (-1)^{1+2}(-4) = 4$

$A_{21} = (-1)^{2+1}(3) = -3 \quad A_{22} = (-1)^{2+2}(2) = 2$

$$\therefore adj\ A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$\text{Now } A(adj\ A) = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \cdot \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{And } (adj\ A)\ A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|A|\ I = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A(adj\ A) = (adj\ A)\ A = |A|\ I$.

Section C

- **Write the answer of the following questions. [Each carries 4 Marks]** **[32]**

9. Find the inverse of the following matric : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

➥ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

$\therefore |A| = 1\ (-\cos^2\alpha - \sin^2\alpha)$

$= -1$

$|A| \neq 0$

$\therefore A^{-1}$ must exists.

Now we find cofactors of given matrix,

$A_{11} = (-1)^{1+1}(-1) = -1$

$A_{12} = (-1)^{1+2}(0) = 0$

$A_{13} = (-1)^{1+3}(0) = 0$

$A_{21} = (-1)^{2+1}(0) = 0$

$A_{22} = (-1)^{2+2}(-\cos\alpha) = -\cos\alpha$

$A_{23} = (-1)^{2+3}(\sin\alpha) = -\sin\alpha$

$A_{31} = (-1)^{3+1}(0) = 0$

$A_{32} = (-1)^{3+2}(\sin\alpha) = -\sin\alpha$

$A_{33} = (-1)^{3+3}(\cos\alpha) = \cos\alpha$

$$\therefore adj\,A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & +\cos\alpha \end{bmatrix}$$

Now $A^{-1} = \frac{1}{|A|}\, adj\ A$

$$= -\frac{1}{1}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & +\cos\alpha \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

10. Find the inverse of the following matric : $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

➥ $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

$\therefore\ |A| = 1(-3 - 0) = -3 \neq 0$

$\therefore\ A^{-1}$ must exists.

$A_{11} = (-1)^{1+1}(-3) = -3$ | $A_{21} = (-1)^{2+1}(0) = 0$

$A_{12} = (-1)^{1+2}(-3) = 3$ | $A_{22} = (-1)^{2+2}(-1) = -1$

$A_{13} = (-1)^{1+3}(-9) = -9$ | $A_{23} = (-1)^{2+3}(2) = -2$

$A_{31} = (-1)^{3+1}(0) = 0$

$A_{32} = (-1)^{3+2}(0) = 0$

$A_{33} = (-1)^{3+3}(3) = 3$

$$\therefore adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

Now $A^{-1} = \frac{1}{|A|}\, adj\ A$

$$\therefore\ A^{-1} = \frac{1}{-3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

11. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

➥ $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

L.H.S. $= A^2 - 5A + 7I$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$= O =$ R.H.S.

Now, $A^2 - 5A + 7I = O$

Multiplying both sides by A^{-1}, we get

$$A^{-1} A^2 - A^{-1} 5A + A^{-1} 7I = A^{-1} O$$

$\therefore\ (A^{-1}A)A - 5(A^{-1}A) + 7(A^{-1}I) = O$

$\therefore\ IA - 5I + 7A^{-1} = O \qquad (\because AA^{-1} = I)$

$\therefore\ A = 5I + 7A^{-1} = O \qquad (\because IA = A)$

$\therefore\ 7A^{-1} = 5I - A$

$$\therefore\ 7A^{-1} = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

12. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, Show that $A^3 - 6A^2 + 5A + 11\,I = O$. Hence, find A^{-1}.

➥ $$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

L.H.S. $= A^3 - 6A^2 + 5A + 11\,I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= O$

$=$ R.H.S.

Now $A^3 - 6A^2 + 5A + 11\,I = O$

Multiplying both sides by A^{-1}, we get

$(A^{-1}A)A^2 - 6(A^{-1}A)A + 5(A^{-1}A) + 11(A^{-1}I) = 0$

$\therefore\ A^2 - 6A + 5\,I + 11A^{-1} = 0 \qquad \because\ A^{-1}A = I,\ A^{-1}I = A^{-1}$

$\therefore\ 11A^{-1} = -A^2 + 6A - 5I$

$$\therefore\ 11A^{-1} = -\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6-5 & -2+6-0 & -1+6-0 \\ 3+6-0 & -8+12-5 & 14-18-0 \\ -7+12-0 & 3-6-0 & -14+18-5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

13. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ Verify the result $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}.

➥ $$A^2 = A \cdot A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-4 & 22-36+18-4 & -21+30-9+0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Now, $A^3 - 6A^2 + 9A - 4I = O$

Multiplying both sides by A^{-1}, we get

$(A^{-1}A)A^2 - 6(AA^{-1})A + 9(A^{-1}A) + 4(A^{-1}I) = O$

$\therefore\ A^2 - 6A + 9I - 4A^{-1} = O$ $\quad (\because\ A^{-1}A = I,\ A^{-1}I = A^{-1})$

$\therefore\ 4A^{-1} = A^2 - 6A + 9I$

$$= -\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ 4A^{-1} = \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

14. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ then find $(AB)^{-1}$.

➥ Here to find $(A \cdot B)^{-1}$ we calculate $B^{-1} \cdot A^{-1}$. As A^{-1} is given we find B^{-1}.

Now $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$\therefore\ |B| = 1(3 - 0) - 2(-1 - 0) - 2(2 - 0)$

$= 3 + 2 - 4$

$= 1 \neq 0$

$\therefore\ B^{-1}$ exists.

Now we find cofactors of matrix B.

$B_{11} = (-1)^{1+1}(3) = 3$ $\quad$ $B_{21} = (-1)^{2+1}(-2) = 2$

$B_{12} = (-1)^{1+2}(-1) = 1$ $\quad$ $B_{22} = (-1)^{2+2}(1) = 1$

$B_{13} = (-1)^{1+3}(2) = 2$ $\quad$ $B_{23} = (-1)^{2+3}(-2) = 2$

$B_{31} = (-1)^{3+1}(+6) = +6$

$B_{32} = (-1)^{3+2}(-2) = 2$

$B_{33} = (-1)^{3+3}(5) = 5$

$$\therefore\ adj\ B = \begin{bmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}\ adj\ B = \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\text{Now } (AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

15. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that (i) $[adj\ A]^{-1} = adj\ (A^{-1})$ (ii) $(A^{-1})^{-1} = A$.

➥ **(i)** $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$|A| = 1(15 - 1) + 2(-10 - 1) + 1(-2 - 3)$

$= 14 - 22 - 5$

$= -13 \neq 0$

$\therefore$ A^{-1} exists.

Now we find cofactors of A,

$A_{11} = (-1)^{1+1}(14) = 14$ $\quad A_{21} = (-1)^{2+1}(-11) = 11$

$A_{12} = (-1)^{1+2}(-11) = 11$ $\quad A_{22} = (-1)^{2+2}(4) = 4$

$A_{13} = (-1)^{1+3}(-5) = -5$ $\quad A_{23} = (-1)^{2+3}(3) = -3$

$A_{31} = (-1)^{3+1}(-5) = -5$

$A_{32} = (-1)^{3+2}(3) = -3$

$A_{33} = (-1)^{3+3}(-1) = -1$

$$\therefore adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A = -\frac{1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

Now suppose $B = adj\ A = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$

$|B| = 14(-4 - 9) - 11(-11 - 15) - 5(-33 + 20)$

$= -182 + 286 + 65$

$= 169 \neq 0$

$B_{11} = (-1)^{1+1}(-13) = -13$

$B_{12} = (-1)^{1+2}(-26) = 26$

$B_{13} = (-1)^{1+3}(-13) = -13$

$B_{31} = (-1)^{3+1}(-13) = -13$

$B_{32} = (-1)^{3+2}(13) = -13$

$B_{33} = (-1)^{3+3}(-65) = -65$

$B_{21} = (-1)^{2+1}(-26) = 26$

$B_{22} = (-1)^{2+2}(-39) = -39$

$B_{23} = (-1)^{2+3}(13) = -13$

$$\therefore adj\ B = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} adj\ B = \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$$\therefore (adj\ A)^{-1} = -\frac{1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & +5 \end{bmatrix} \quad ....(1)$$

Let $C = A^{-1} = -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$

$$adj \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix} (\because adj\ B)$$

$$\therefore adj\ C = adj(A^{-1}) = -\frac{1}{169} adj \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$= -\frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$$adj(A^{-1}) = -\frac{1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 11 \\ 1 & 1 & 5 \end{bmatrix} \quad ....(2)$$

From (1) and (2) we get,

$(adj\ A)^{-1} = adj\ (A^{-1})$

➥ **(ii)** Let $B = A^{-1} = -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$

$$\therefore B = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

$$|B| = -\frac{14}{13}\left(-\frac{4}{169}-\frac{9}{169}\right)+\frac{11}{13}\left(-\frac{11}{169}-\frac{15}{169}\right)+\ \frac{5}{13}\left(-\frac{33}{169}+\frac{20}{169}\right)$$

$$= -\frac{14}{13}\left(-\frac{13}{169}\right)+\frac{11}{13}\left(-\frac{26}{169}\right)+\frac{5}{13}\left(-\frac{13}{169}\right)$$

$$= +\frac{14}{169}-\frac{22}{169}-\frac{5}{169}$$

$$= -\frac{13}{169}$$

$$= -\frac{1}{13}$$

$$adj\ B = adj\begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

$$= \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & 65 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

$$(B^{-1}) = \frac{1}{|B|}\ adj\ B$$

$$= \frac{1}{-\frac{1}{13}} \times \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$= A$

Hence, $(A^{-1})^{-1} = A$

16. Solve the following system of linear equations

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z} = 1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z} = 2$$

➥ Given equations can be written in matrix form as, AX = B

Where $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, $X = \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$|A| = 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$

$= 150 + 330 + 720$

$= 1200 \neq 0$

$\therefore$ A^{-1} exists.

$A_{11} = (-1)^{1+1}(75) = 75$ | $A_{21} = (-1)^{2+1}(-150) = 150$

$A_{12} = (-1)^{1+2}(-110) = 110$ | $A_{22} = (-1)^{2+2}(-100) = -100$

$A_{13} = (-1)^{1+3}(72) = 72$ $A_{23} = (-1)^{2+3}(0) = 0$

$A_{31} = (-1)^{3+1}(75) = 75$

$A_{32} = (-1)^{3+2}(-30) = 30$

$A_{33} = (-1)^{3+3}(-24) = -24$

$$\therefore adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$= \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Now $AX = B$

$\therefore$ $A^{-1}AX = A^{-1}B$

$\therefore$ $X = A^{-1}B$ $(\because A^{-1}A = I,\ IX = X)$

$$\therefore \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

On comparing the corresponding elements we get,

$$\therefore \frac{1}{x} = \frac{1}{2},\ \frac{1}{y} = \frac{1}{3},\ \frac{1}{z} = \frac{1}{5}$$

$\therefore$ $x = 2,\ y = 3,\ z = 5.$