OPEN STUDENT FOUNDATION

CHAPTER 6 | Std 12 : MATHS | Date : 23/02/24

PRACTICE SHEET DAY 5

Section A

- **Write the answer of the following questions. [Each carries 2 Marks]** **[6]**

1. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm ?

2. Find the intervals in given function is strictly increasing or decreasing : $10 - 6x - 2x^2$

3. Find both the maximum value and the minimum value of $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$.

Section B

- **Write the answer of the following questions. [Each carries 3 Marks]** **[21]**

4. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

5. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

6. Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function of x throughout its domain.

7. Find the local maxima and local minima of the given function. Find also the local maximum and the local minimum value of the given function : $f(x) = x^3 - 6x^2 + 9x + 15$

8. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

9. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

10. Find the intervals in which the function f given by $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is (i) increasing (ii) decreasing.

Section C

- **Write the answer of the following questions. [Each carries 4 Marks]** **[40]**

11. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

12. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

13. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

14. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

15. A wire of length 28 m. is to be cut into two pieces. One of the pieces is to be made into a square

and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum ?

16. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

17. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

18. Find the points at which the function f given by $f(x) = (x - 2)^4 (x + 1)^3$ has

(i) local maxima,

(ii) local minima,

(iii) point of inflexion.

19. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

20. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4\pi}{27} h^3 \tan^2\alpha$.

Section A

- **Write the answer of the following questions. [Each carries 2 Marks]** **[6]**

1. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm ?

➥ Let the side of a cube be x cm.

$\therefore$ Its volume $V = x^3$ and its surface area $S = 6x^2$.

Given that $\frac{dV}{dt} = 8\ cm^3/sec.$

Now $V = x^3$

$\therefore \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

$\therefore 8 = 3x^2 \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = \frac{8}{3x^2}$ ....(i)

Now $S = 6x^2$

$\therefore \frac{dS}{dt} = 12x \frac{dx}{dt}$

$\therefore \frac{dS}{dt} = 12x \times \frac{8}{3x^2}$ ($\because$ From (i))

$\therefore \frac{dS}{dt} = \frac{32}{x}$

$\therefore \frac{dS}{dt} = \frac{32}{12}$ ($\because x = 12$ cm)

$\therefore \frac{dS}{dt} = \frac{8}{3}$ cm^2/sec.

Hence, the surface area of a cube increases at the rate of $\frac{8}{3} cm^2/sec.$ when the length of an edge is 12 cm.

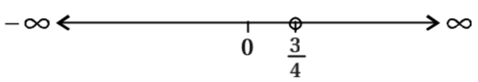
2. Find the intervals in given function is strictly increasing or decreasing : $10 - 6x - 2x^2$

➥ $f(x) = 10 - 6x - 2x^2$

$\therefore f'(x) = -6 - 4x$

$f'(x) = 0 \Rightarrow x = -\frac{3}{2}$

$x = -\frac{3}{2}$ divides a number line into two disjoint intervals.



In $\left(-\frac{3}{2}, \infty\right)$, $f'(x) < 0$

$\therefore$ In $\left(-\frac{3}{2}, \infty\right)$ interval, the given function $f(x)$ is strictly decreasing.

In $\left(-\infty, -\frac{3}{2}\right)$, $f'(x) > 0$

$\therefore$ $\left(-\infty, -\frac{3}{2}\right)$ interval, the given function $f(x)$ is strictly increasing.

3. Find both the maximum value and the minimum value of $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval [0, 3].

➥ $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25,\ x \in [0, 3]$

$\therefore f'(x) = 12x^3 - 24x^2 + 24x - 48$

$= 12\ [x^3 - 2x^2 + 2x - 4]$

$= 12\ [x^2(x - 2) + 2(x - 2)]$

$= 12\ (x^2 + 2)\ (x - 2)$

For maximum or minimum value $f'(x) = 0$

$\therefore 12(x^2 + 2)(x - 2) = 0$

$\therefore x = 2$ ($\because x^2 = -2$ is not possible.)

Now $f(2) = 3(2)^4 - 8(2)^3 + 12(2)^2 - 48(2) + 25$

$= 48 - 64 + 48 - 96 + 25$

$= -39$

$x \in [0, 3]$ $\therefore f(0) = 3(0)^4 - 8(0)^3 + 12(0)^2 - 48(0) + 25$

$= 25$

and $f(3) = 3(3)^4 - 8(3)^3 + 12(3)^2 - 48(3) + 25$

$= 243 - 216 + 108 - 144 + 25$

$= 16$

$\therefore f$ has maximum value 25 at $x = 0$.

and minimum value –39 at $x = 2$.

Section B

● **Write the answer of the following questions.** [**Each carries 3 Marks**] **[21]**

4. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

➥ Let r be the radius and V be volume of the balloon.

$\frac{dV}{dt} = 900$ cm^3/sec.

Volume of the balloon $V = \frac{4}{3}\pi r^3$

$$\therefore \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\cdot\frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt}$$

$$\therefore 900 = 4\pi (15)^2 \frac{dr}{dt} \quad (\because \text{ Given } r = 15 \text{ cm})$$

$$\therefore \frac{dr}{dt} = \frac{900}{4\pi \times 15 \times 15} = \frac{1}{\pi} \text{ cm/sec.}$$

Hence, the radius of the balloon is increases at the rate of $\frac{1}{\pi}$ cm/sec. when the radius is 15 cm.

5. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

➥ $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta, \quad \theta \in \left[0, \frac{\pi}{2}\right]$

$$y = f(\theta) = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$$

$$\therefore f'(\theta) = \frac{(2+\cos\theta)(4\cos\theta) - (4\sin\theta)(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2} \quad (\because \cos^2\theta + \sin^2\theta = 1)$$

$$= \frac{8\cos\theta + 4 - 4 - 4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

Now $f'(\theta) = 0 \Rightarrow \cos\theta(4-\cos\theta) = 0$

$\therefore \cos\theta = 0$ or $4 - \cos\theta = 0$

$\therefore \theta = \frac{\pi}{2} \quad (\because \cos\theta = 4$ is not possible)

$\therefore$ For $0 < \theta < \frac{\pi}{2}$, $f'(\theta) > 0$ ($\because$ Because $\cos\theta$ is positive in first quadrant and $0 < \cos\theta < 1$)

$\therefore 0 < \theta < \frac{\pi}{2}$, the function is an increasing function.

$\therefore f$ is an increasing function in $\left(0, \frac{\pi}{2}\right)$.

6. Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function of x throughout its domain.

➥ $y = \log(1+x) - \frac{2x}{2+x}, \; x > -1$

$\log(1 + x)$ is defined only for $x > -1$

$$y = f(x) = \log(1 + x) - \frac{2x}{2 + x}$$

Differentiate with respect to x,

$$\therefore f'(x) = \frac{1}{1 + x} - \frac{(2 + x)(2) - 2x(1)}{(2 + x)^2}$$

$$= \frac{1}{1 + x} - \frac{4 + 2x - 2x}{(2 + x)^2}$$

$$= \frac{(2 + x)^2 - 4(1 + x)}{(1 + x)(2 + x)^2}$$

$$= \frac{4 + 4x + x^2 - 4 - 4x}{(1 + x)(2 + x)^2}$$

$$= \frac{x^2}{(1 + x)(2 + x)^2}$$

when $f'(x) = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

Case : I If $-1 < x < 0$ then $f'(x) > 0$ as $x + 1 > 0$ and for each $x \in R$, $x^2 > 0$ and $(2 + x)^2$.

$\therefore f$ is an increasing function in the interval $(-1, 0)$.

Case : II If $x > 0$ then $f'(x) > 0$

$\therefore f$ is an increasing function in $(0, \infty)$.

Hence, f is an increasing function of x throughout its domain.

7. Find the local maxima and local minima of the given function. Find also the local maximum and the local minimum value of the given function : $f(x) = x^3 - 6x^2 + 9x + 15$

➥ $\boldsymbol{f(x) = x^3 - 6x^2 + 9x + 15}$

$f(x) = x^3 - 6x^2 + 9x + 15$

$\therefore f'(x) = 3x^3 - 12x + 9$

$f''(x) = 6x - 12$

For critical poits,

$f'(x) = 0 \therefore 3x^2 - 12x + 9 = 0$

$\therefore 3(x^2 - 4x + 3) = 0$

$\therefore 3(x - 3)(x - 1) = 0$

$\therefore x = 3, 1$

$f''(3) = 6(3) - 12 = 18 - 12 = 6 > 0$

$f''(1) = 6(1) - 12 = 6 - 12 = -6 < 0$

$\therefore f(x)$ has local maxima at $x = 1$ and local minima at $x = 3$.

Local maximum value of $f(x)$,

$f(1) = (1)^3 - 6(1)^2 + 9(1) + 15$

$= 1 - 6 + 9 + 15$

$= 19$

Local minimum value of $f(x)$,

$f(3) = (3)^3 - 6(3)^2 + 9(3) + 15$

$= 27 - 54 + 27 + 15$

$= 15$

8. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

➥ Here x and y are given positive numbers such that $x + y = 60$

$\therefore\ y = 60 - x$ ....(i)

Now $f(x) = xy^3$

$= x(60 - x)^3$ (From (i))

$\therefore\ f'(x) = x\left(3(60 - x)^2 \cdot (-1)\right) + (60 - x)^3$

$= -3x(60 - x)^2 + (60 - x)^3$ ....(ii)

$\therefore\ f''(x) = -3\{(60 - x)^2(1) + x \cdot 2(60 - x)^2(-1)\} + 3(60 - x)^2(-1)$

$\therefore\ f''(x) = -3(60 - x)^2 - 6x(60 - x)^2 - 3(60 - x)^2$..(iii)

Now maximum/minimum value we take $f'(x) = 0$

$\therefore\ -3x(60 - x)^2 + (60 - x)^3 = 0$

$\therefore\ (60 - x)^2\ \{-3x + 60 - x\} = 0$

$\therefore\ 60 - x = 0$ OR $60 - 4x = 0$

$\therefore\ x = 60$ which is not possible as sum of two numbers is 60.

We take $60 - 4x = 0$

$\therefore\ 4x = 60$

$\therefore\ x = 15$

$\therefore\ f''(15) = -3(60-15)^2 - 6(15)(60-15)^2 - 3(60-15)^2$

$= -3(45)^2 - 90(45)^2 - 3(45)^2$

$= (45)^2\ \{-3 - 90 - 3\}$

$= (45)^2\ (-96)$

$f''(15) < 0$

$\therefore\ f$ is maximum for $x = 15$.

$\therefore$ One part is $x = 15$ and other part is $y = 60 - x$.

i.e. $y = 60 - 15 = 45$

$\therefore$ Required numbers are 15 and 45.

9. Show that the function given by $f(x) = \dfrac{\log x}{x}$ has maximum at $x = e$.

➥ $f(x) = \dfrac{\log x}{x}$

$$\therefore f'(x) = \frac{x \cdot \left(\frac{1}{x}\right) - \log x \cdot (1)}{x^2}$$

$$= \frac{1 - \log x}{x^2}$$

$$\therefore f''(x) = \frac{x^2 \cdot \left(-\frac{1}{x}\right) - (1 - \log x) \cdot 2x}{x^4}$$

$$= \frac{-x - 2x + 2x \log x}{x^4}$$

$$= \frac{-3 + 2\log x}{x^3}$$

For maximum value take $f'(x) = 0$

$$\therefore \frac{1 - \log x}{x^2} = 0$$

$\therefore \quad 1 - \log x = 0 \qquad (\because \; x \neq 0)$

$\therefore \quad \log x = 1$

$\therefore \quad x = e$

$$\therefore f''(e) = \frac{-3 + 2\log e}{e^3}$$

$$= \frac{-3 + 2}{e^3}$$

$$= -\frac{1}{e^3} < 0$$

$\therefore f(x) = \dfrac{\log x}{x}$ has maximum at $x = e$.

10. Find the intervals in which the function f given by $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is (i) increasing (ii) decreasing.

➥ $$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$$

$$= \frac{4\sin x - x(2 + \cos x)}{2 + \cos x}$$

$$= \frac{4\sin x}{2 + \cos x} - x$$

$$\therefore f'(x) = \frac{(2 + \cos x)(4\cos x) - 4\sin x(-\sin x)}{(2 + \cos)^2} - 1$$

$$= \frac{8\cos x + 4\cos^2 x + 4\sin^2 x}{(2 + \cos)^2} - 1$$

$$= \frac{8\cos x + 4 - (2 + \cos x)^2}{(2 + \cos)^2}$$

$$= \frac{8\cos x + 4 - 4 - 4\cos x - \cos^2 x}{(2 + \cos)^2}$$

$4\cos x - \cos^2 x$

$$= \frac{}{(2 + \cos)^2}$$

$$= \frac{\cos x\,(4 - \cos x)}{(2 + \cos)^2}$$

Now $-1 \le \cos x \le 1$

$\therefore\ 4 - \cos x > 0$ Also $(2 + \cos x)^2 > 0$

If $\cos x > 0$ then $f'(x) > 0$ and

If $\cos x < 0$ then $f'(x) < 0$.

$\therefore\ \cos x > 0 \Rightarrow 0 < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$

$\therefore$ In the interval $0 < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, $f(x)$ is strictly increasing function.

And if $\cos x < 0$ then $\frac{\pi}{2} < x < \frac{3\pi}{2}$

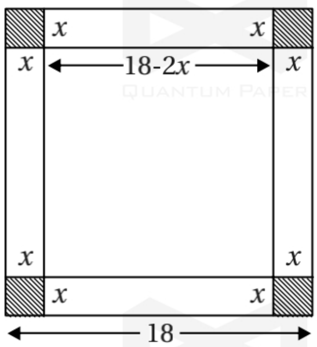
$\therefore$ In the interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$, $f(x)$ is strictly decreasing function.

Section C

- **Write the answer of the following questions. [Each carries 4 Marks]** **[40]**

11. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

➥ Let the length of the side of a square cutting from each corner of the square piece tin be x cm.



A box without top has,

l = Length = $18 - 2x$ cm

b = Breadth = $18 - 2x$ cm

h = Height = x cm

Volume of the box V = lbh

$\therefore\ V = (18 - 2x)(18 - 2x)x \text{ cm}^3$

$\therefore\ V = x(18 - 2x)^2$

$$\therefore\ \frac{dV}{dx} = (18 - 2x)^2 + x \cdot 2(18 - 2x) \cdot (-2)$$

$$= (18 - 2x)^2 - 4x\,(18 - 2x)$$

$$= (18 - 2x)\,[18 - 2x - 4x]$$

$= (18 - 2x)\ (18 - 6x)$

$\therefore \dfrac{d^2V}{dx^2} = (18 - 2x)(-6) + (18 - 6x)(-2)$

$= -108 + 12x - 36 + 12x$

$= 24x - 144$

For maximum value $\dfrac{dV}{dx} = 0$

$\therefore (18 - 2x)(18 - 6x) = 0$

$\therefore x = 9,\ x = 3$

$\left.\dfrac{d^2V}{dx^2}\right|_{x=9} = 24(9) - 144 > 0$

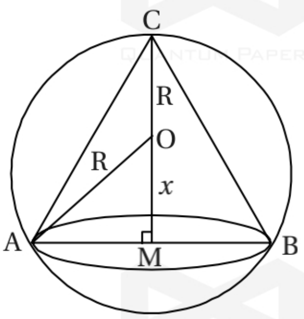
$\left.\dfrac{d^2V}{dx^2}\right|_{x=3} = 24(3) - 144 < 0$

$\therefore$ For $x = 3$, the volume of the box is maximum.

$\therefore$ The side of the square to be cut off is $x = 3$ cm.

12. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\dfrac{8}{27}$ of the volume of the sphere.

➥ The volume of largest cone CAB is inscribed in a sphere of radius R. The axis of the cone is diameter of the sphere. O is the centre of the sphere.



$\therefore$ OA = OB = OC = R

Let OM = x

In ΔOMA, $R^2 = x^2 + AM^2$

$\therefore AM = \sqrt{R^2 - x^2}$

$\therefore$ Radius of the base of the cone,

$r = AM = \sqrt{R^2 - x^2}$

Height of the cone h = CM = CO + OM = R + x

Now volume of the cone $V = \dfrac{1}{3}\pi r^2 h$

$\therefore V = \dfrac{1}{3}\pi\left(\sqrt{R^2 - x^2}\right)^2 (R + x)$

$= \frac{1}{3}\pi(R^2 - x^2)(R + x)$

$= \frac{1}{3}\pi(R^3 + R^2x - Rx^2 - x^3)$

$\therefore \frac{dV}{dx} = \frac{1}{3}\pi(R^2 - 2Rx - 3x^2)$ $(\because$ R = constant)

and $\frac{d^2V}{dx^2} = \frac{1}{3}\pi(-2R - 6x)$

For maximum volume $\frac{dV}{dx} = 0$

$\therefore \frac{1}{3}\pi(R^2 - 2Rx - 3x^2) = 0$

$\therefore R^2 - 2Rx - 3x^2 = 0$

$\therefore R^2 - 3Rx + Rx - 3x^2 = 0$

$\therefore (R + x)(R - 3x) = 0$

$\therefore R = 3x$ $(\because R + x \neq 0)$

$\therefore x = \frac{R}{3}$

and $\left.\frac{d^2V}{dx^2}\right|_{x=\frac{R}{3}} = \frac{1}{3}\pi(-2R - 2R) = -\frac{4\pi R}{3} < 0$

$\therefore$ The volume of the cone is maximum for $x = \frac{R}{3}$.

Volume of the cone $V = \frac{1}{3}\pi(R^2 - x^2)(R + x)$

$= \frac{1}{3}\pi\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right)$

$= \frac{1}{3}\pi\left(\frac{8R^2}{9}\right)\left(\frac{4R}{3}\right)$

$= \frac{8}{27}\left(\frac{4\pi R^3}{3}\right)$

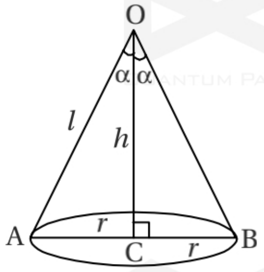
$= \frac{8}{27}$

(Volume of the sphere)

Hence, the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

13. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

➥ Let r be the radius, h be the height, l be the slant height and α be the semi vertical angle of the cone.



The slant height l is given.

So, l = constant

From figure $\sin\alpha = \frac{AC}{OA} = \frac{r}{l}$

$$\therefore\ r = l\sin\alpha$$

and $\cos\alpha = \frac{OC}{OA} = \frac{h}{l}$

$$\therefore\ h = l\cos\alpha$$

Volume of the cone $V = \frac{1}{3}\pi r^2 h$

$$\therefore\ V = \frac{1}{3}\pi\ (l\ \sin\alpha)^2\ (l\ \cos\alpha)$$

$$\therefore\ V = \frac{1}{3}\pi\ l^3\ \sin^2\alpha\ \cos\alpha$$

$$\therefore\ \frac{dV}{d\alpha} = \frac{1}{3}\pi\ l^3\ [2\sin\alpha\ \cos^2\alpha - \sin^3\alpha]$$

$$= \frac{1}{3}\pi\ l^3 \sin\alpha\ [2\cos^2\alpha - \sin^2\alpha] \quad \text{......(i)}$$

$$\therefore\ \frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi l^3 \left[\cos\alpha(2\cos^2\alpha - \sin^2\alpha) + \sin\alpha(-4\cos\alpha\ \sin\alpha - 2\sin\alpha\ \cos\alpha)\right]$$

$$= \frac{1}{3}\pi l^3 [2\cos^3\alpha - \cos\alpha\sin^2\alpha - 6\cos\alpha\ \sin^2\alpha]$$

$$= \frac{1}{3}\pi l^3\ [2\cos^3\alpha - 7\cos\alpha\ \sin^2\alpha]$$

$$= \frac{1}{3}\pi l^3\ \cos^3\alpha\ [2 - 7\tan^2\alpha] \quad \text{......(ii)}$$

For maximum volume $\frac{dV}{d\alpha} = 0$

$$\therefore\ \frac{1}{3}\pi l^3\ \sin\alpha\ [2\cos^2\alpha - \sin^2\alpha] = 0 \quad \text{(From (i))}$$

$$\therefore\ 2\cos^2\alpha - \sin^2\alpha = 0$$

$$\therefore\ \tan^2\alpha = 2$$

$$\therefore\ \tan\alpha = \sqrt{2}$$

Now $\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{3}}$

(ii) $\therefore\ \frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi l^3\ \cos^3\alpha\ (2 - 7\tan^2\alpha)$

$$\therefore \left.\frac{d^2V}{d\alpha^2}\right|_{\tan\alpha=\sqrt{2}} = \frac{1}{3}\pi l^3\left(\frac{1}{\sqrt{3}}\right)^3 (2-7(2))$$

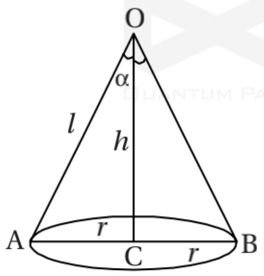
$$= \frac{\pi l^3}{9\sqrt{3}}(2-14)$$

$$= \frac{-4\pi l^3}{3\sqrt{3}} < 0$$

Hence, volume of the cone is maximum at $\tan\alpha = \sqrt{2}$. i.e. The semi vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}(\sqrt{2})$.

14. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

➥ Let r is the radius, h height l slant height and α is the semi vertical angle of the cone.



The surface area of the cone, $S = \pi r^2 + \pi rl$

Now S is given $\Rightarrow$ S = constant

$$l = \frac{S - \pi r^2}{\pi r} \quad \text{............(i)}$$

Volume of the cone $V = \frac{1}{3}\pi r^2 h$

$$\therefore V^2 = \frac{1}{9}\pi^2 r^4 h^2$$

$$= \frac{1}{9}\pi^2 r^4 (l^2 - r^2)$$

$$= \frac{1}{9}\pi^2 r^4\left[\left(\frac{S-\pi r^2}{\pi r}\right)^2 - r^2\right] \quad (\because \text{From (i)})$$

$$= \frac{1}{9}\pi^2 r^4\left[\frac{(S-\pi r^2)^2 - \pi^2 r^4}{\pi^2 r^4}\right]$$

$$= \frac{1}{9}\pi^2 r^4\left[\frac{S^2 - 2\pi S r^2}{\pi^2 r^4}\right]$$

$$= \frac{1}{9}r^2 S\left[S - 2\pi r^2\right]$$

$$= \frac{1}{9}S\left[Sr^2 - 2\pi r^4\right]$$

Take $Z = V^2 \Rightarrow Z = \frac{1}{9}S\left[Sr^2 - 2\pi r^4\right]$

$\therefore \dfrac{dZ}{dr} = \dfrac{1}{9} S \left[2 Sr - 8\pi r^3\right]$

and $\dfrac{d^2Z}{dr^2} = \dfrac{1}{9} S \left[2S - 24\pi r^2\right]$

For maximum volume $\dfrac{dZ}{dr} = 0$

$\therefore \dfrac{1}{3} S\ (2Sr - 8\pi r^3) = 0$

$\therefore 2Sr - 8\pi r^3 = 0$

$\therefore S = 4\pi r^2 = 0$

$\therefore r^2 = \dfrac{S}{4\pi}$

$\therefore \dfrac{d^2Z}{dr^2} = \dfrac{1}{9} S\ [2S - 24\pi r^2]$

$$\therefore \left.\dfrac{d^2Z}{dr^2}\right|_{r^2 = \frac{S}{4\pi}} = \dfrac{1}{9} S \left[2S = 24\pi\left(\dfrac{S}{4\pi}\right)\right]$$

$$= \dfrac{1}{9} S\ [-4S]$$

$$= \dfrac{-4S^2}{9} < 0$$

$\therefore$ The volume of the cone is maximum for $S = 4\pi r^2$

$S = 4\pi r^2$

$\therefore \pi rl + \pi r^2 = 4\pi r^2$

$\therefore \pi rl = 3\pi r^2$

$\therefore l = 3r$

$\therefore \dfrac{r}{l} = \dfrac{1}{3}$

From figure, $\sin\alpha = \dfrac{AC}{OA} = \dfrac{r}{l}$

$\therefore \sin\alpha = \dfrac{1}{3}$

$\therefore \alpha = \sin^{-1}\left(\dfrac{1}{3}\right)$

Hence, the semi vertical angle of right circular cone of given surface and maximum volume is $\sin^{-1}\left(\dfrac{1}{3}\right)$.

15. A wire of length 28 m. is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum ?

➥ A wire of length 28m is to be cut into two pieces.

Let the length of one piece be x m.

$\therefore$ The length of another piece will be $(28 - x)$m.

From a wire of length x met, a circle with radius r is made and from a wire of length $(28 - x)$m. a square is made.

Circumference of a circle = x

$\therefore\ 2\pi r = x \Rightarrow r = \dfrac{x}{2\pi}$

Perimeter of the square = $28 - x$

$\therefore$ Length of side of the square $= \dfrac{28 - x}{4}$

Now Area of circle + Area of square

$A = \pi r^2 + (\text{side})^2$

$\therefore\ A = \pi\left(\dfrac{x}{2\pi}\right)^2 + \left(\dfrac{28 - x}{4}\right)^2$

$\therefore\ A = \dfrac{x^2}{4\pi} + \dfrac{(28 - x)^2}{16}$

$\dfrac{dA}{dx} = \dfrac{2x}{4\pi} + \dfrac{2(28 - x)(-1)}{16}$

$= \dfrac{x}{2\pi} - \dfrac{(28 - x)}{8}$

For the combined area to become minumum $\dfrac{dA}{dx} = 0$

$\therefore\ \dfrac{x}{2\pi} - \dfrac{(28 - x)}{8} = 0$

$\therefore\ 4x = 28\pi - \pi x$

$\therefore\ x(4 + \pi) = 28\pi$

$\therefore\ x = \dfrac{28\pi}{4 + \pi}$ m.

$\dfrac{d^2A}{dx^2} = \dfrac{1}{2\pi} + \dfrac{1}{8} > 0$

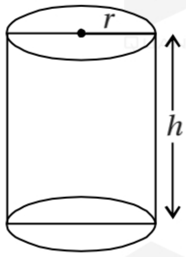
$\therefore$ For $x = \dfrac{28\pi}{4 + \pi}$, the combined area of the square and the circle is minimum.

$28 - x = 28 - \dfrac{28\pi}{4 + \pi} = \dfrac{112}{4 + \pi}$ m.

Hence, the length of the two pieces of wire are $\dfrac{28\pi}{4 + \pi}$ m and $\dfrac{112}{4 + \pi}$ m.

16. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

➥ Let h be the height of the cylinder and r be the radius of its base. The curved surface area of the cylinder is S and its volume is V.



$\therefore$ $S = 2\pi r^2 + 2\pi rh$

Now S is given $\Rightarrow$ S = constant

$\therefore h = \dfrac{S - 2\pi r^2}{2\pi r}$ ....(i)

Volume $V = \pi r^2 h$

$\therefore V = \pi r^2 \left(\dfrac{S - 2\pi r^2}{2\pi r}\right)$ ($\because$ From (i))

$= \dfrac{Sr - 2\pi r^3}{2}$

$= \dfrac{1}{2} Sr - \pi r^3$

$\therefore \dfrac{dV}{dr} = \dfrac{d}{dr}\left(\dfrac{1}{2} Sr - \pi r^3\right)$

$= \dfrac{1}{2} S - 3\pi r^2$

$\dfrac{d^2V}{dr^2} = \dfrac{d}{dr}\left(\dfrac{dV}{dr}\right) = \dfrac{d}{dr}\left(\dfrac{1}{2} S - 3\pi r^2\right) = -6\pi r$

For maximum volume $\dfrac{dV}{dr} = 0$

$\therefore \dfrac{1}{2} S - 3\pi r^2 = 0$

$\therefore$ $S = 6\pi r^2$ and $\dfrac{d^2V}{dr^2} = -6\pi r < 0$ ($\because r > 0$)

But $S = 2\pi r^2 + 2\pi rh$

$\therefore$ $6\pi r^2 = 2\pi r^2 + 2\pi rh$

$\therefore$ $4\pi r^2 = 2\pi rh$

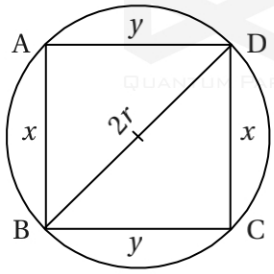
$\therefore$ $h = 2r$

$\therefore$ $h = 2r$ = diameter of the base

Hence, the right circular cylinder of given surface, the maximum volume occurs, when its height is equal to the diameter of the base.

17. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

➥ A rectangle □ ABCD is inscribed in a circle. Let side AB = CD = x unit and AD = BC = y unit.



Let radius of the circle = r

$\therefore$ BD = $2r$

In ΔABD, According to pythagours theorem, $AB^2 + AD^2 = BD^2$

$\therefore x^2 + y^2 = (2r)^2$

$\therefore y = \sqrt{4r^2 - x^2}$ ....(1)

Area of a rectangle ABCD, A = xy

$\therefore A = x \cdot \sqrt{4r^2 - x^2}$ ($\because$ From (i))

$$\therefore \frac{dA}{dx} = \sqrt{4r^2 - x^2} - \frac{x \cdot 2x}{2\sqrt{4r^2 - x^2}}$$

$$= \sqrt{4r^2 - x^2} - \frac{x^2}{\sqrt{4r^2 - x^2}}$$

$$= \frac{4r^2 - x^2 - x^2}{\sqrt{4r^2 - x^2}}$$

$$\therefore \frac{dA}{dx} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

$$\therefore \frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2} \cdot (-4x) - (4r^2 - 2x^2) \cdot \frac{(-2x)}{2\sqrt{4r^2 - x^2}}}{(4r^2 - x^2)}$$

$$= \frac{-4x(4r^2 - x^2) + x(4r^2 - 2x^2)}{(4r^2 - x^2)^{\frac{3}{2}}}$$

$$= \frac{-16r^2x - 4x^3 + 4x^2x - 2x^3}{(4r^2 - x^2)^{\frac{3}{2}}}$$

$$= \frac{2x^3 - 12r^2x}{(4r^2 - x^2)^{\frac{3}{2}}}$$

For maximum/minimum value take $\frac{dA}{dx} = 0$

$$\therefore \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0$$

$\therefore 4r^2 - 2x^2 = 0 \Rightarrow x = \sqrt{2}r$

$$\left.\frac{d^2A}{dx^2}\right|_{x=\sqrt{2}r} = \frac{2(\sqrt{2}r)^3 - 12r^2(\sqrt{2}r)}{(4r^2 - 2r^2)^{\frac{3}{2}}}$$

$$= \frac{4\sqrt{2}r^3 - 12\sqrt{2}r^3}{(2r^2)^{\frac{3}{2}}} = \frac{-8\sqrt{2}\,r^3}{2\sqrt{2}\,r^3}$$

$$= -4 < 0$$

$\therefore$ For $x = \sqrt{2}r$, Area is maximum.

$$x = \sqrt{2}r \Rightarrow y = \sqrt{4r^2 - x^2}$$

$$= \sqrt{4r^2 - 2r^2}$$

$$= \sqrt{2}r$$

$\therefore x = y = \sqrt{2}r$

Here the sides of a rectangle are equal the hence it is a square.

$\therefore$ Of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

18. Find the points at which the function f given by $f(x) = (x-2)^4\ (x+1)^3$ has
(i) local maxima,
(ii) local minima,
(iii) point of inflexion.

➡ $f(x) = (x-2)^4\ (x+1)^3$

$\therefore f'(x) = (x-2)^4\ 3(x+1)^2 + 4(x-2)^3(x+1)^3$

$= (x-2)^3(x+1)^2\ [3(x-2) + 4(x+1)]$

$= (x-2)^3(x+1)^2\ [7x-2]$

For critical point, $f'(x) = 0$

$\therefore (x-2)^3(x+1)^2\ (7x-2) = 0$

$\therefore x - 2 = 0,\ x + 1 = 0,\ 7x - 2 = 0$

$\therefore x = 2,\ x = -1,\ x = \frac{2}{7}$

At $x = 2$:

When $x < 2$, $f'(x) = (-ve)\ (+ve)\ (+ve) = (-ve)$

When $x > 2$, $f'(x) = (+ve)\ (+ve)\ (+ve) = (+ve)$

Thus $f'(x)$ change its sign from $(-ve)$ to $(+ve)$.

$\therefore f(x)$ has local minimum value at $x = 2$.

At $x = -1$:

When $x < -1$ then, $f'(x) = (-ve)\ (+ve)\ (-ve) = (+ve)$

If $x > -1$ then, $f'(x) = (-ve)\ (+ve)\ (-ve) = (+ve)$

$\therefore$ Sign of $f'(x)$ does not change for $x = -1$.

$\therefore$ Function $f(x)$ has no maximum or minimum value for $x = -1$.

For $x = \frac{2}{7} = 0.28$:

If $x < 0.28$ then, $f'(x) = (-ve)\ (+ve)\ (-ve) = +ve$

And if $x > 0.28$ then, $f'(x) = (-ve)\ (+ve)\ (+ve) = -ve$

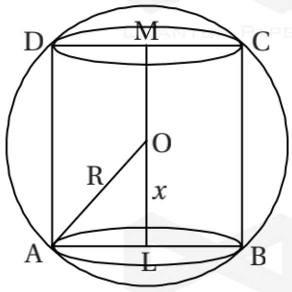
Hence for $x = \frac{2}{7}$, $f'(x)$ changes $+ve$ to $-ve$.

$\therefore$ $f(x)$ has local maximum at $x = \frac{2}{7}$.

$\therefore$ Hence, $f(x)$ has local maxima at $x = \frac{2}{7}$, local minimum at $x = 2$ and point of inflexion at $x = -1$.

19. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

➥ The cylinder if inscribed in a sphere of radius R. For maximum volume of cylinder, the axis of cylinder is a diameter of the sphere.



Let O be the centre of the sphere.

OA = R, Let OL = x

Where ML is axis of the cylinder.

ML = 20L = $2x$

ΔOLA is a right angle triangle.

$\therefore\ OA^2 = AL^2 + OL^2$

$\therefore\ R^2 = AL^2 + x^2$

$\therefore\ AL = \sqrt{R^2 - x^2}$

Height of the cylinder ABCD = R

$\therefore\ h = ML = 2x$

Radius of the base $r = AL = \sqrt{R^2 - x^2}$

Where R = constant

The volume of the cylinder, $V = \pi r^2 h$

$\therefore\ V = \pi\ (R^2 - x^2)\ 2x$

$\therefore\ V = 2\pi\ (R^2 x - x^3)$

$\therefore\ \frac{dV}{dx} = 2\pi\ (R^2 - 3x^2)$

and $\frac{d^2V}{dx^2} = 2\pi\ (0 - 6x) = -12\ \pi x$

For maximum volume, $\frac{dV}{dx} = 0$

$\therefore\ 2\pi\ (R^2 - 3x^2) = 0$

$\therefore\ R^2 - 3x^2 = 0$

$\therefore\ x = \frac{R}{\sqrt{3}}$

$$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{R}{\sqrt{3}}} = \frac{-12\pi R}{\sqrt{3}} < 0$$

$\therefore$ For $x = \frac{R}{\sqrt{3}}$ the volume of the cylinder is maximum.

Height of the cylinder $h = 2x = \frac{2R}{\sqrt{3}}$

Maximum volume $V = \pi\ (R^2 - x^2)2x$

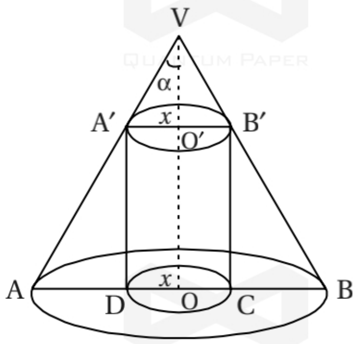
$$\therefore V = \pi\left(R^2 - \frac{R^2}{3}\right)\frac{2R}{\sqrt{3}} \quad \left(\because x = \frac{R}{\sqrt{3}}\right)$$

$$= \frac{4\pi R^3}{3\sqrt{3}}$$

Hence, the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$ and its volume is $\frac{4\pi R^3}{3\sqrt{3}}$.

20. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4\pi}{27} h^3 \tan^2\alpha$.

➥ The cylinder A'B'CD is inscribed in a cone VAB. Height of the cone VO = h is given. The semi vertical angle of the cone is α.



$\therefore\ \angle AVO = \alpha$

Let the radius of the cylinder be x.

$\therefore$ A'O' = OD = x

$\overline{AB} \parallel \overline{A'B'}$

$\therefore\ \angle AVO = \angle A'VO' = \alpha$

ΔA'O'V is a right angle triangle.

$\therefore \tan\alpha = \frac{A'O'}{VO'} = \frac{x}{VO'}$

$\therefore$ VO' $= \dfrac{x}{\tan\alpha} = x \cot\alpha$

Height of the cylinder OO' = VO – VO'

$= h - x \cot\alpha$

The volume of the cylinder V $= \pi(\text{OD})^2$ (OO')

$= \pi x^2 (h - x \cot\alpha)$

$= \pi hx^2 - \pi x^3 \cot\alpha$

$$\frac{dV}{dx} = 2\pi hx - 3\pi x^2 \cot\alpha$$

and $\dfrac{d^2V}{dx^2} = 2\pi h - 6\pi x \cot\alpha$

For maximum volume $\dfrac{dV}{dx} = 0$

$\therefore 2\pi hx - 3\pi x^2 \cot\alpha = 0$

$\therefore \pi x (2h - 3x \cot\alpha) = 0$

But $x \neq 0$

$\therefore 2h - 3x \cot\alpha = 0$

$\therefore x = \dfrac{2h}{3\cot\alpha} = \dfrac{2h}{3}\tan\alpha$

$$\left.\frac{d^2V}{dx^2}\right|_{x = \frac{2h\tan\alpha}{3}} = 2\pi h - 6\pi\left(\frac{2h\tan\alpha}{3}\right)\cot\alpha$$

$= 2\pi h - 4\pi h$

$= -2\pi h < 0$

$\therefore$ The volume of the cylinder inscribed in a cone has greatest volume when $x = \dfrac{2h\tan\alpha}{3}$.

Height of the cylinder

OO' $= h - x\cot\alpha$

$= h - \dfrac{2h\tan\alpha}{3}\cot\alpha = h - \dfrac{2h}{3}$

$= \dfrac{1}{3}h$

$= \dfrac{1}{3}$ (Height of the cone)

The greatest volume of the cylinder

$V = \pi x^2 (h - x\cot\alpha)$

$$= \pi\left(\frac{4h^2\tan^2\alpha}{9}\right)\left(h - \frac{2h\tan\alpha\cos\alpha}{3}\right)$$

$$= \frac{4\pi h^2\tan^2\alpha}{9}\left(\frac{h}{3}\right)$$

$$= \frac{4}{27} \pi h^3 \tan^2 \alpha$$