OPEN STUDENT FOUNDATION

CHAPTER 9 — Std 12 : MATHS — Date : 26/02/24

PRACTICE SHEET DAY 8

Section A

- **Choose correct answer from the given options. [Each carries 1 Mark]** **[4]**

1. The order of the differential equation $2x^2 \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$ is ....

(A) 2 (B) 1 (C) 0 (D) not defined

2. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$ is ....

(A) 3 (B) 2 (C) 1 (D) not defined

3. The number of arbitary constants in the general solution of a differential equation of fourth order are ....

(A) 0 (B) 2 (C) 3 (D) 4

4. The number of arbitary constants in the particular solution of a differential equation of third order are ....

(A) 3 (B) 2 (C) 1 (D) 0

Section B

- **Write the answer of the following questions. [Each carries 1 Mark]** **[1]**

1. Determine order and degree (if defined) of differential equation : $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

Section C

- **Write the answer of the following questions. [Each carries 2 Marks]** **[8]**

2. For the differential equation, Find the general solution : $(e^x + e^{-x})\, dy - (e^x - e^{-x})\, dx = 0$

3. For the differential equation, Find the general solution : $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

4. Verify that the given function (implicit or explicit) is a solution of the corresponding differential equation : $y = e^x(a \cos x + b \sin x)$: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

5. Verify that the given function (implicit or explicit) is a solution of the corresponding differential equation : $y = ae^x + be^{-x} + x^2$: $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

Section D

- **Write the answer of the following questions. [Each carries 3 Marks]** **[12]**

6. For the differential equation, Find the general solution : $e^x \tan y\, dx + (1 - e^x) sec^2 y\, dy = 0$

7. For differential equation, find the general solution : $\frac{dy}{dx} + \frac{y}{x} = x^2$

8. For differential equation, find the general solution : $(x + 3y^2)\frac{dy}{dx} = y \quad (y > 0)$

9. Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1\ (x \neq 0)$.

Section E

- **Write the answer of the following questions. [Each carries 4 Marks]** **[32]**

10. show that the given differential equation is homogeneous and solve it : $x^2 \frac{dy}{dx} = x^2 - 2y^2 + xy$

11. show that the given differential equation is homogeneous and solve it : $\left\{x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right\} y\, dx = \left\{y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right\} x\, dy$

12. show that the given differential equation is homogeneous and solve it : $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$

13. show that the given differential equation is homogeneous and solve it : $(1 + e^{\frac{x}{y}})\, dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy = 0$

14. show that the given differential equation is homogeneous and solve it : $x \frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$

15. For differential equation, find the general solution : $x \frac{dy}{dx} + y - x + xy \cot x = 0\ (x \neq 0)$

16. Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of differential equation $(x^3 - 3xy^2)\, dx = (y^3 - 3x^2y)$dy, where c is a parameter.

17. Show that the general solution of the differential equation $\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$ is given by $(x + y + 1) = A\ (1 - x - y - 2xy)$, where A is parameter.

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PRACTICE SHEET DAY 8

Section A

- **Choose correct answer from the given options. [Each carries 1 Mark]** **[4]**

1. The order of the differential equation $2x^2 \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$ is ....

(A) 2 (B) 1 (C) 0 (D) not defined

Ans. (A) 2

➥ The order of the given differential equation is 2.

2. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$ is ....

(A) 3 (B) 2 (C) 1 (D) not defined

Ans. (D) not defined

➥ The given differential equation is not a polynomial equation in its derivatives so its degree is not defined.

3. The number of arbitary constants in the general solution of a differential equation of fourth order are ....

(A) 0 (B) 2 (C) 3 (D) 4

Ans. (D) 4

➥ The order of the differential equation is 4. so the number of arbitary constants in the general solution of a differential equation is 4.

∴ Alternate (D)

4. The number of arbitary constants in the particular solution of a differential equation of third order are ....

(A) 3 (B) 2 (C) 1 (D) 0

Ans. (D) 0

➥ The particular solution of a differential equation is free from arbitary constants. so the number of arbitary constants is 0.

∴ Alternate (D)

Section B

- **Write the answer of the following questions. [Each carries 1 Mark]** **[1]**

1. Determine order and degree (if defined) of differential equation : $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

➥ The heighest order derivative present in the given differential equation is $\frac{d^2s}{dt^2}$. so its degree is 2. The heighest power raised to $\frac{d^2s}{dt^2}$ is one. so its degree is one.

Section C

- **Write the answer of the following questions. [Each carries 2 Marks]** **[8]**

2. For the differential equation, Find the general solution : $(e^x + e^{-x})\ dy - (e^x - e^{-x})\ dx = 0$

➥ $(e^x + e^{-x})\ dy - (e^x - e^{-x})\ dx = 0$

$\therefore\ (e^x + e^{-x})\ dy = (e^x - e^{-x})\ dx$

$$\therefore\ dy = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}\ dx$$

Integrating both sides, we get

$$\int dy = \int\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) dx$$

$$\therefore \int dy = \int \frac{\frac{d}{dx}(e^x + e^{-x})}{e^x + e^{-x}}\ dx \quad \left(\text{Use } \int \frac{f'(x)}{f(x)}\ dx = \log|f(x)| + c\right)$$

$\therefore\ y = \log|e^x + e^{-x}| + c$

which is the general solution of the given differential equation.

3. For the differential equation, Find the general solution : $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

➥ $$\frac{dy}{dx} = (1 + x^2)(1 + y^2)$$

$$\therefore \frac{dy}{1 + y^2} = (1 + x^2)\ dx$$

Integrating both sides, we get

$$\int \frac{dy}{1 + y^2} = \int (1 + x^2)\ dx$$

$$\therefore \tan^{-1} y = x + \frac{x^3}{3} + c$$

which is the general solution of the given differential equation.

4. Verify that the given function (implicit or explicit) is a solution of the corresponding differential equation : $y = e^x (a \cos x + b \sin x)$: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

➥ $y = e^x (a \cos x + b \sin x)$ ... (1)

Differentiating w.r. to x, we have

$$\therefore \frac{dy}{dx} = e^x(-a \sin x + b \cos x) + e^x (a \cos x + b \sin x)$$

$$\therefore \frac{dy}{dx} = e^x(-a \sin x + b \cos x) + y \quad \text{... (2)} \quad (\because \text{ From (1)})$$

Again Differentiating w.r. to x, we have

$$\frac{d^2y}{dx^2} = e^x(-a \cos x - b \sin x) + e^x (-a \sin x + b \cos x) + \frac{dy}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = -e^x (a \cos x + b \sin x) + \frac{dy}{dx} - y + \frac{dy}{dx} \quad (\because \text{ From (2)})$$

$$\therefore \frac{d^2y}{dx^2} = -y + \frac{2dy}{dx} - y \qquad (\because \text{ From } (1))$$

$$\therefore \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Therefore, the given function is a solution of given differential equation.

5. Verify that the given function (implicit or explicit) is a solution of the corresponding differential equation : $y = ae^x + be^{-x} + x^2$: $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

➥ $xy = ae^x + be^{-x} + x^2$ ... (1)

Differentiating eq. (1) w.r. to x, we have

$$x\frac{dy}{dx} + y = ae^x - be^{-x} + 2x \qquad \text{... (2)}$$

Again differentiating w.r. to x, we get

$$x\frac{dy^2}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = ae^x + be^{-x} + 2 \qquad \text{... (3)}$$

From eg. (1) $\Rightarrow ae^x + be^{-x} = xy - x^2$

This value is substitue in (3), we get

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{2dy}{dx} - xy + x^2 - 2 = 0$$

Therefore, the given function is a solution of given differential equation.

Section D

● **Write the answer of the following questions. [Each carries 3 Marks]** **[12]**

6. For the differential equation, Find the general solution : $e^x \tan y\, dx + (1 - e^x)\, sec^2y\, dy = 0$

➥ $e^x \tan y\, dx + (1 - e^x)\, sec^2y\, dy = 0$

$$\therefore \frac{e^x}{1-e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

Integrating the equation, we have

$$\int \frac{e^x}{1-e^x}\, dx + \int \frac{\sec^2 y}{\tan y}\, dy = 0$$

$$\therefore -\int \frac{\frac{d}{dx}(1-e^x)}{1-e^x}\, dx + \int \frac{\frac{d}{dy}(\tan y)}{\tan y}\, dy = 0$$

$$\therefore -\log|1 - e^x| + \log|\tan y| = \log c \quad \left(\text{Use } \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right)$$

$$\therefore \log\left|\frac{\tan y}{1-e^x}\right| = \log c$$

$$\therefore \frac{\tan y}{1-e^x} = c$$

$\therefore$ $\tan y = c(1 - e^x)$. which is the general solution of the given differential equation.

7. For differential equation, find the general solution : $\frac{dy}{dx} + \frac{y}{x} = x^2$

➥ $\frac{dy}{dx} + \frac{y}{x} = x^2$ ... (1)

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$

where $P = \frac{1}{x}$ and $Q = x^2$

$\therefore$ I.F. $= e^{\int P\,dx} = e^{\int \frac{1}{x}\,dx}$

$= e^{\log_e x} = x$

Multiplying eq. (1) by x, we have

$x\frac{dy}{dx} + x\cdot\left(\frac{y}{x}\right) = x\cdot x^2$

$\therefore$ $x\frac{dy}{dx} + y = x^3$

$\therefore$ $\frac{d}{dx}(xy) = x^3$

Integrating both sides with respect to x,

$xy = \int x^3\,dx$

$\therefore$ $xy = \frac{x^4}{4} + c$

Which is a general solution of the given differential equation.

8. For differential equation, find the general solution : $(x + 3y^2)\frac{dy}{dx} = y$ $(y > 0)$

➥ $(x + 3y^2)\frac{dy}{dx} = y$

$\therefore$ $\frac{(x+3y^2)}{y} = \frac{dx}{dy}$

$\therefore$ $\frac{dx}{dy} - \frac{x}{y} = 3y$ ... (1)

$\therefore$ This is a linear equation of the

$\frac{dx}{dy} + P_1\,x = Q_1$.

where $P_1 = -\frac{1}{y}$ and $Q_1 = 3y$

I.F. $= e^{\int P_1\,dy}$

$= e^{\int -\frac{1}{y}\,dy}$

$= e^{-\log y} = \frac{1}{y}$

Multiplying eq. (1) by $\frac{1}{y}$, we get

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = \frac{1}{y}(3y)$$

$$\therefore \frac{d}{dy}\left(\frac{1}{y}\cdot x\right) = 3$$

Integrating w.r.t. to y, we have

$$\frac{x}{y} = \int 3dy$$

$$\therefore \frac{x}{y} = 3y + c$$

$$\therefore x = 3y^2 + cy$$

Which is the general solution of the given differential equation.

9. Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1 \ (x \neq 0)$.

➥ $$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1$$

$$\therefore \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} + \frac{1}{\sqrt{x}}y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \qquad \dots (1)$$

This is a linear equation of the type $\frac{dy}{dx} + \text{P}y = \text{Q}$

Where $\text{P} = +\frac{1}{\sqrt{x}}$, $\text{Q} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Integrating Factor (I.F.) $= e^{\int \text{P}\,dx}$

$$= e^{\int + \frac{1}{\sqrt{x}}\,dx}$$

$$= e^{2\sqrt{x}}$$

Multiplying eq. (1) by $e^{2\sqrt{x}}$, we get

$$\frac{dy}{dx}e^{2\sqrt{x}} + \frac{1}{\sqrt{x}}\cdot e^{2\sqrt{x}}\cdot y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \times e^{2\sqrt{x}}$$

$$\therefore \frac{d}{dx}\left(y\cdot e^{2\sqrt{x}}\right) = \frac{1}{\sqrt{x}}$$

Integrating, we get

$$y\cdot e^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\,dx$$

$$\therefore y\cdot e^{2\sqrt{x}} = 2\sqrt{x} + c$$

Which is the solution of the given differential equation.

Section E

- **Write the answer of the following questions. [Each carries 4 Marks]** **[32]**

10. show that the given differential equation is homogeneous and solve it : $x^2 \dfrac{dy}{dx} = x^2 - 2y^2 + xy$

➥ $x^2 \dfrac{dy}{dx} = x^2 - 2y^2 + xy$

$\therefore \dfrac{dy}{dx} = \dfrac{x^2 - 2y^2 + xy}{x^2}$ ... (1)

Let $F(x, y) = \dfrac{x^2 - 2y^2 + xy}{x^2}$ $\therefore F(\lambda x, \lambda y) = \dfrac{\lambda^2 x^2 - 2\lambda^2 y^2 + \lambda^2 xy}{\lambda^2 x^2}$

$= \dfrac{\lambda^2 (x^2 - 2y^2 + xy)}{\lambda^2 x^2} = \lambda^0 F(x, y)$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

Take $y = vx$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substitute this value in eq. (1), we get

$v + x\dfrac{dv}{dx} = \dfrac{x^2 - 2v^2x^2 + vx^2}{x^2}$

$\therefore x\dfrac{dv}{dx} = 1 - 2v^2 + v - v$

$\therefore x\dfrac{dv}{dx} = 1 - 2v^2 = -(2v^2 - 1)$

$\therefore \dfrac{dv}{2v^2 - 1} = -\dfrac{dx}{x}$

Integrating both sides, we get

$\int \dfrac{dv}{2v^2 - 1} = \int -\dfrac{dx}{x}$

$\therefore \dfrac{1}{2}\int \dfrac{dv}{v^2 - \left(\dfrac{1}{\sqrt{2}}\right)^2} = -\int \dfrac{dx}{x}$

$\therefore \dfrac{1}{2} \times \dfrac{1}{2\left(\dfrac{1}{\sqrt{2}}\right)} \log \left| \dfrac{v - \dfrac{1}{\sqrt{2}}}{v + \dfrac{1}{\sqrt{2}}} \right| = -\log x + c$

Now replace $v = \dfrac{y}{x}$,

$\therefore \dfrac{1}{2\sqrt{2}} \log \left| \dfrac{\dfrac{y}{x} - \dfrac{1}{\sqrt{2}}}{\dfrac{y}{x} + \dfrac{1}{\sqrt{2}}} \right| = -\log x + c$

$$\therefore \frac{1}{2\sqrt{2}} \log \left|\frac{\sqrt{2}y - x}{\sqrt{2}y + x}\right| = -\log x + c$$

$$\therefore \frac{1}{2\sqrt{2}} \log \left|\frac{x + \sqrt{2}y}{x - \sqrt{2}y}\right| = -\log x + c$$

Which is a required general solution.

11. show that the given differential equation is homogeneous and solve it : $\left\{x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right\} y\, dx$ $= \left\{y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right\} x\, dy$

➡ $$\left\{x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right\} y\, dx = \left\{y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right\} x\, dy$$

$$\therefore \frac{dy}{dx} = \frac{y\left[x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right]}{x\left[y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right]} \quad \ldots (1)$$

Let $F(x, y) = \dfrac{y\left[x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right]}{x\left[y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right]}$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda y\left[\lambda x \cos\left(\frac{y}{x}\right) + \lambda y \sin\left(\frac{y}{x}\right)\right]}{\lambda x\left[\lambda y \sin\left(\frac{y}{x}\right) - \lambda x \cos\left(\frac{y}{x}\right)\right]}$$

$$= \frac{\lambda^2 y\left[x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right]}{\lambda^2 x\left[y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right]}$$

$$= \lambda^0 F(x, y)$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitute this value in eq. (1), we have

$$v + x\frac{dv}{dx} = \frac{vx\,[x \cos v + vx \sin v]}{x\,[vx \sin v - x \cos v]}$$

$$\therefore x\frac{dv}{dx} = \frac{v[\cos v + v \sin v]}{v \sin v - \cos v} - v$$

$$\therefore x\frac{dv}{dx} = \frac{2v \cos v}{v \sin v - \cos v}$$

$$\therefore \left(\frac{v \sin v - \cos v}{2v \cos v}\right) dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(\frac{v\sin v-\cos v}{2v\cos v}\right)dv=\int\frac{dx}{x}$$

$$\therefore \frac{1}{2}\int\tan v\,dv-\frac{1}{2}\int\frac{1}{v}\,dv=\int\frac{dx}{x}$$

$$\therefore \frac{1}{2}\log|\sec v|-\frac{1}{2}\log|v|=\log|x|+\log c_1$$

Now replace $v=\frac{y}{x}$,

$$\frac{1}{2}\log\left|\sec\frac{y}{x}\right|-\frac{1}{2}\log\left|\frac{y}{x}\right|=\log|x|+\log c_1$$

$$\therefore -\frac{1}{2}\log\left|\cos\frac{y}{x}\right|-\frac{1}{2}\log\left|\frac{y}{x}\right|-\log|x|=\log c_1$$

$$\therefore \log\left|\cos\frac{y}{x}\right|+\log\left|\frac{y}{x}\right|+2\log x=-2\log c_1$$

$$\therefore \cos\left(\frac{y}{x}\right)\times\frac{y}{x}\times x^2=c$$

$$\therefore xy\cos\left(\frac{y}{x}\right)=c$$

12. show that the given differential equation is homogeneous and solve it : $(x^2+xy)\,dy=(x^2+y^2)\,dx$

➥ $(x^2+xy)\,dy=(x^2+y^2)\,dx$

$$\therefore \frac{dy}{dx}=\frac{x^2+y^2}{x^2+xy} \quad \ldots (1)$$

Let $F(x,y)=\dfrac{x^2+y^2}{x^2+xy}$

$$\therefore F(\lambda x,\lambda y)=\frac{\lambda^2x^2+\lambda^2y^2}{\lambda^2x^2+\lambda^2xy}$$

$$=\frac{\lambda^2(x^2+y^2)}{\lambda^2(x^2+xy)}=\lambda^0F(x,y)$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

Take $y=vx$,

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dy}{dx}$$

Substitute this value in eq. (1), we get

$$v+x\frac{dv}{dx}=\frac{x^2+v^2x^2}{x^2+vx^2}$$

$$\therefore x\frac{dv}{dx}=\frac{1+v^2}{1+v}-v$$

$$\therefore x\frac{dv}{dx}=\frac{1-v}{1+v}$$

$$\therefore \quad x\frac{dv}{dx} = -\left(\frac{v-1}{v+1}\right)$$

$$\therefore \left(\frac{v+1}{v-1}\right)dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(\frac{v+1}{v-1}\right)dv = -\int\frac{dx}{x}$$

$$\therefore \int\left(\frac{v-1+2}{v-1}\right)dv = -\int\frac{dx}{x}$$

$$\therefore \int\left(1+\frac{2}{v-1}\right)dv = -\int\frac{dx}{x}$$

$$\therefore v + 2\log(v-1) = -\log x + c_1$$

Replace $v = \frac{y}{x}$, we get

$$\frac{y}{x} + 2\log\left(\frac{y}{x}-1\right) = -\log x + c_1$$

$$\therefore \frac{y}{x} + 2\log(y-x) - 2\log x + \log x = c_1$$

$$\therefore \frac{y}{x} + 2\log(y-x) - \log x = c_1$$

$$\therefore 2\log(y-x) - \log x = c_1 - \frac{y}{x}$$

$$\therefore \log\left[\frac{(y-x)^2}{x}\right] = c_1 - \frac{y}{x}$$

$$\therefore \frac{(y-x)^2}{x} = e^{c_1 - \frac{y}{x}} = e^{c_1}\cdot e^{-\frac{y}{x}} = c\, e^{-\frac{y}{x}}$$

$$\therefore (y-x)^2 = x\cdot c\cdot e^{-\frac{y}{x}}$$

$$\therefore (x-y)^2 = c\,x\,e^{-\frac{y}{x}} \text{ where, } c = e^{c_1}$$

Which is the general solution of the given differential equation.

13. show that the given differential equation is homogeneous and solve it : $(1 + e^{\frac{x}{y}})\,dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

➥ $(1 + e^{\frac{x}{y}})\,dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

$$\therefore \frac{dx}{dy} = \frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}$$

$$F(x, y) = \frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{\frac{x}{\ }}$$ is a homogeneous function.

$1 + e^{y}$

$\therefore$ The given differential equation is a homogeneous differential equation.

Take $\frac{x}{y} = v \Rightarrow x = vy$

Differentiate w.r. to y, we have

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substitute this value in eq. (1), we get

$$v + y\frac{dv}{dy} = \frac{-e^{v}(1-v)}{1+e^{v}}$$

$$\therefore y\frac{dv}{dy} = \frac{-e^{v} + v\,e^{v}}{1+e^{v}} - v$$

$$\therefore y\frac{dv}{dy} = \frac{-e^{v} - v}{1+e^{v}} = -\frac{(e^{v}+v)}{1+e^{v}}$$

$$\therefore \left(\frac{1+e^{v}}{e^{v}+v}\right)dv = -\frac{dy}{y}$$

Integrating both sides, we get

$$\int\left(\frac{1+e^{v}}{v+e^{v}}\right)dv = \int\frac{-dy}{y}$$

$$\therefore \log(v + e^{v}) = -\log y + \log c \quad \left(\because \frac{d}{dv}(v+e^{v}) = 1+e^{v}\right)$$

$$\therefore \log(v + e^{v}) + \log y = \log c$$

$$\therefore y(v + e^{v}) = c$$

Now replace $v = \frac{x}{y}$,

$$y\left(\frac{x}{y} + e^{\frac{x}{y}}\right) = c$$

$$\therefore x + y\,e^{\frac{x}{y}} = c$$

Which is a required general solution.

14. show that the given differential equation is homogeneous and solve it : $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

➥ $$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

$$\therefore \frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x} \quad \ldots (1)$$

Let $F(x, y) = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$

$$\therefore\ F(\lambda x, \lambda y) = \frac{\lambda y - \lambda x \sin\left(\frac{y}{x}\right)}{\lambda x} = \lambda^0\ F\ (x,\ y)$$

Thus, F(x, y) is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

Take $y = vx,\ \frac{dy}{dx} = v + x\ \frac{dv}{dx}$

Substitute this value in eq. (1), we get

$$v + x\ \frac{dv}{dx} = \frac{vx\ - x \sin v}{x}$$

$$\therefore\ x\frac{dv}{dx} = v - \sin v - v$$

$$\therefore\ \frac{dv}{\sin v} = -\ \frac{dx}{x}$$

Integrating both sides, we get

$$\int \operatorname{cosec} v\ dv = -\int \frac{dx}{x}$$

$$\therefore\ \log|\operatorname{cosec}\ v - \cot v| = -\ \log\ x + \log\ c$$

$$\therefore\ \operatorname{cosec}\ v\ - \cot\ v = \frac{c}{x}$$

Now replace $v = \frac{y}{x}$

$$\operatorname{cosec}\ \frac{y}{x} - \cot\frac{y}{x} = \frac{c}{x}$$

$$\therefore\ \frac{1 - \cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{c}{x}$$

$$\therefore\ x\left[1 - \cos\left(\frac{y}{x}\right)\right] = c \sin\left(\frac{y}{x}\right)$$

Which is a required general solution.

15. For differential equation, find the general solution : $x\ \frac{dy}{dx} + y - x + xy\ \cot\ x = 0\ (x \neq 0)$

➥ $x\ \frac{dy}{dx} + y - x + xy\ \cot\ x = 0$

$$\therefore\ x\ \frac{dy}{dx} + y\,(1 + x \cot x) = x$$

$$\therefore\ \frac{dy}{dx} + y\left(\frac{1 + x\ \cot x}{x}\right) = 1 \qquad \dots\ (1)$$

This is a linear equation of the form $\frac{dy}{dx}$ + Py = Q.

where $P = \frac{1 + x \cot x}{x}$ and $Q = 1$

$$\text{I.F.} = e^{\int P\, dx}$$

$$= e^{\int\left(\frac{1}{x} + \cot x\right)dx}$$

$= e^{\log x + \log \sin x}$

$= e^{\log (x \sin x)}$

$= x \sin x$

Multiplying eq. (1) by $x \sin x$,

$$x \sin x \frac{dy}{dx} + x \sin x \cdot y \left(\frac{1 + x\cot x}{x}\right) = x \sin x$$

$$\therefore\ x \sin x \frac{dy}{dx} + (\sin x + x \cos x) = x \sin x$$

$$\therefore\ \frac{d}{dx}\left((x \sin x) \cdot y\right) = x \sin x$$

Integrating w.r. to x, we get

$$\therefore\ (x \sin x) \cdot y = \int x \sin x \cdot dx$$

$$\therefore\ (x \sin x)\, y = x \int \sin x \cdot dx - \int\left(\frac{d}{dx}(x) \int \sin x\, dx\right) dx$$

$$\therefore\ (x \sin x)\ y = -x \cos x - \int -\cos x\ dx$$

$$\therefore\ (x \sin x)\ y = -x \cos x + \sin x + c$$

$$\therefore\ y = -\cot x + \frac{1}{x} + \frac{c}{x \sin x}$$

Which is the general solution of the given differential equation.

16. Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of differential equation $(x^3 - 3xy^2)\, dx = (y^3 - 3x^2y)dy$, where c is a parameter.

➥ $(x^3 - 3xy^2)\ dx = (y^3 - 3x^2y)\ dy$

$$\therefore\ \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2 y} \quad \dots (1)$$

It is a homogeneous differential equation.

Take $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute this value in eq. (1), we have

$$v + x\frac{dv}{dx} = \frac{x^3 - 3x^3 v^2}{x^3v^3 - 3x^3 v}$$

$$\therefore\ x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v$$

$$\therefore\ x\frac{dv}{dx} = \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v}$$

$$\therefore\ x\frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v}$$

$$\therefore\ x\frac{dv}{dx} = -\left(\frac{v^4 - 1}{v^3 - 3v}\right)$$

$$\therefore\ \left(\frac{v^3 - 3v}{v^4 - 1}\right) dv = -\frac{dx}{x}$$

Integrating both sides, we have

$$\int\left(\frac{v^3-3v}{v^4-1}\right)dv=-\int\frac{dx}{x}$$

$$\therefore \int\frac{v^3}{v^4-1}\,dv-3\int\frac{v}{v^4-1}\,dv=-\int\frac{dx}{x}$$

$$\therefore \frac{1}{4}\int\frac{4v^3}{v^4-1}-\frac{3}{2}\int\frac{2v}{(v^2)^2-1}\,dv=-\int\frac{dx}{x}$$

$$\therefore \frac{1}{4}\log|v^4-1|-\frac{3}{2}\times\frac{1}{2}\log\left|\frac{v^2-1}{v^2+1}\right|=-\log x+\log k$$

Now replace $v=\frac{y}{x}$,

$$\frac{1}{4}\log\left|\frac{y^4}{x^4}-1\right|-\frac{3}{4}\log\left|\frac{\frac{y^2}{x^2}-1}{\frac{y^2}{x^2}+1}\right|=-\log x+\log k$$

$$\therefore \frac{1}{4}\log\left|\frac{y^4-x^4}{x^4}\right|-\frac{3}{4}\log\left|\frac{y^2-x^2}{y^2+x^2}\right|=\log\frac{k}{x}$$

$$\therefore \log\left[\frac{y^4-x^4}{x^4}\right]^{\frac{1}{4}}-\log\left[\frac{y^2-x^2}{y^2+x^2}\right]^{\frac{3}{4}}=\log\frac{k}{x}$$

$$\therefore \left(\frac{y^4-x^4}{x^4}\right)^{\frac{1}{4}}\times\left(\frac{y^2+x^2}{y^2-x^2}\right)^{\frac{3}{4}}=\frac{k}{x}$$

$$\therefore \frac{(y^2-x^2)^{\frac{1}{4}}(y^2+x^2)^{\frac{1}{4}}}{x}\times\frac{(y^2+x^2)^{\frac{3}{4}}}{(y^2-x^2)^{\frac{3}{4}}}=\frac{k}{x}$$

$$\therefore \frac{(y^2+x^2)}{(y^2-x^2)^{\frac{1}{2}}}=k$$

$$\therefore (y^2+x^2)=k\,(y^2-x^2)^{\frac{1}{2}}$$

Squaring both sides, we get

$$(y^2+x^2)^2=k^2\,(y^2-x^2)$$

$$\therefore (y^2-x^2)=\frac{(y^2+x^2)^2}{k^2}$$

$$\therefore x^2-y^2=-\frac{1}{k^2}(x^2+y^2)^2$$

$$\therefore x^2-y^2=c\,(x^2+y^2)^2 \text{ where } c=-\frac{1}{k^2}$$

17. Show that the general solution of the differential equation $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$ is given by $(x+y+1) = A\,(1-x-y-2xy)$, where A is parameter.

➥ $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

$\therefore \frac{dy}{y^2+y+1} + \frac{dx}{x^2+x+1}$

Integrating, we get

$$\int \frac{dy}{y^2+y+1} + \int \frac{dx}{x^2+x+1} = 0$$

$$\Rightarrow \int \frac{dy}{y^2+y+\frac{1}{4}+\frac{3}{4}} + \int \frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}} = 0$$

$$\Rightarrow \int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = 0$$

$$\Rightarrow \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = c$$

$$\Rightarrow \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2} c$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}} + \frac{2x+1}{\sqrt{3}}}{1-\left(\frac{2y+1}{\sqrt{3}}\right)\left(\frac{2x+1}{\sqrt{3}}\right)}\right] = \frac{\sqrt{3}}{2} c$$

$$\Rightarrow \frac{2y+1+2x+1}{3-(2y+1)(2x+1)} \times \frac{3}{\sqrt{3}} = \tan\left(\frac{\sqrt{3}}{2} c\right)$$

$$\Rightarrow \frac{\sqrt{3}\,(2x+2y+2)}{3-4xy-2x-2y-1} = \tan\left(\frac{\sqrt{3}}{2} c\right)$$

$$\Rightarrow \frac{2\sqrt{3}\,(x+y+1)}{2\,(1-x-y-2xy)} = \tan\left(\frac{\sqrt{3}}{2} c\right)$$

$$\Rightarrow x+y+1 = \frac{1}{\sqrt{3}} \tan\left(\frac{\sqrt{3}}{2} c\right)(1-x-y-2xy)$$

$$\Rightarrow (x+y+1) = A(1-x-y-2xy)$$

Where $A = \frac{1}{\sqrt{3}} \tan\left(\frac{\sqrt{3}}{2} c\right)$ is a parameter.

Which is the required general solution.